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SUBMANIFOLDS WITH PARALLEL MEAN CURVATURE VECTOR

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0. Introduction

J. Simons [5] has recently proved a formula which gives the Laplacian of the square of the length of the second fundamental form, and applied the formula to the study of minimal hypersurfaces in the sphere (see also [1], [2]).

K. Nomizu and B. Smyth [4] have obtained a formula of the same type for a hypersurface immersed with constant mean curvature in a space of constant sectional curvature, and derived a new formula for the Laplacian of the square of the length of the second fundamental form, in which the sectional curvature of the hypersurface appears. Using this new formula, they determined hypersurfaces of nonnegative sectional curvature and constant mean curvature immersed in the Euclidean space or in the sphere under the additional condition that the square of the length of the second fundamental form is constant.

The purpose of the present paper is to generalize Nomizu-Smyth formulas to the case of general submanifolds and to use the formulas to study submanifolds, immersed in a space of constant curvature, whose normal bundle is locally parallelizable and mean curvature vector field is parallel in the normal bundle.

1. Preliminaries

Let there be given an n-dimensional connected submanifold M^n immersed in an m-dinensional Riemannian manifold M^m $(1 < n < m)$ with the metric tensor G, whose components are G_{ji} with respect to local coordinates $\{\xi^h\}$, (Riemannian manifolds we discuss are assumed to be differentiable and of class C^∞.) and suppose that the local expression of the submanifold M^n in M^m is

$$\xi^h = \xi^h(\eta^a) , \tag{1.1}$$

where $\{\eta^a\}$ are local coordinates in M^n. (Submanifolds we discuss are always assumed to be differentiable, of class C^∞ and connected. The indices h, i, j, k, l run over the range $\{1, \cdots, m\}$ and the indices a, b, c, d, e over the range $\{1, \cdots, n\}$. The summation convention is used with respect to these systems of indices.) Differentiate (1.1) and put

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$$B_b{}^h = \partial_b \xi^h , \qquad \partial_b = \partial/\partial\eta^b , \tag{1.2}$$

which is, for each fixed index b, a local vector field tangent to M^n. These local vector fields $B_b{}^h$ spann the tangent plane of M^n at each point of M^n. We denote by $C_x{}^h$ $m-n$ mutually orthogonal local unit vector fields normal to M^n. (The indices x, y, z run over the range $\{n+1, \cdots, m\}$ and the summation convention is used with respect to this system of indices.)

If we denote by g the metric tensor on M^n induced from the metric tensor G of M^m, then for the components of g we have

$$g_{cb} = G_{ji} B_c{}^j B_b{}^i . \tag{1.3}$$

The contravariant components of g are denoted by g^{cb}, i,e., $g_{ce}g^{eb} = \delta_c^b$.

Denoting by $\{_j{}^h{}_i\}$ and $\{_c{}^a{}_b\}$ the Christoffel symbols formed with G_{ji} and g_{cb} respectively, we put

$$\nabla_c B_b{}^h = \partial_c B_b{}^h + \{_j{}^h{}_i\} B_c{}^j B_b{}^i - \{_c{}^a{}_b\} B_a{}^h , \tag{1.4}$$

which is the *van der Waerden-Bortolotti covariant derivative* of $B_b{}^h$. From (1.2) and (1.4) we then have

$$\nabla_c B_b{}^h = \nabla_b B_c{}^h . \tag{1.5}$$

For tensor fields on M^n, ∇_c is the operator of covariant differentiation with respect to $\{_c{}^a{}_b\}$. The van der Waerden-Bortolotti covariant differentiation ∇_c is extended to tensor fields of mixed type, say $T_b{}^a{}_i{}^h$, on M^n in such a way that

$$\begin{aligned} \nabla_c T_b{}^a{}_i{}^h = \partial_c T_b{}^a{}_i{}^h &+ \{_j{}^h{}_k\} B_c{}^j T_b{}^a{}_i{}^k - \{_j{}^k{}_i\} B_c{}^j T_b{}^a{}_k{}^h \\ &+ \{_c{}^a{}_e\} T_b{}^e{}_i{}^h - \{_c{}^e{}_b\} T_e{}^a{}_i{}^h . \end{aligned}$$

Thus we have

$$\begin{aligned} \nabla_d \nabla_c B_b{}^h = \partial_d(\nabla_c B_b{}^h) &+ \{_j{}^h{}_i\} B_d{}^j \nabla_c B_b{}^i \\ &- \{_d{}^e{}_c\} \nabla_e B_b{}^h - \{_d{}^e{}_b\} \nabla_c B_e{}^h . \end{aligned} \tag{1.6}$$

It is easily verified that for any fixed indices b and c, $\nabla_c B_b{}^h$ is normal to M^n, and hence that

$$\nabla_c B_b{}^h = h_{cb}{}^x C_x{}^h , \tag{1.7}$$

where $h_{cb}{}^x$ satisfies, due to (1.5),

$$h_{cb}{}^x = h_{bc}{}^x . \tag{1.8}$$

The $h_{cb}{}^x$ is, for each fixed index x, a local tensor field of type (0,2) of M^n and called the *second fundamental tensor* of the submanifold M^n relative to the unit normal $C_x{}^h$. Equations (1.7) are the *equations of Gauss* for the submanifold M^n.

If we denote by g^* the metric tensor induced on the normal bundle $\mathcal{N}(M^n)$ of M^n from the metric tensor G of M^n, then we have, for the components of g^* relative to the frame $\{C_x{}^h\}$,

$$g_{yx} = G_{ji} C_y{}^j C_x{}^i = \delta_{yx} ,$$

because $C_x{}^h$ are orthonormal. The contravariant components of g^* are given by $g^{yx} = \delta^{yx}$, since $g_{yz} g^{zx} = \delta_y^x$.

If we put

$$h^x = g^{cb} h_{cb}{}^x / n , \qquad h^2 = g_{yx} h^y h^x \qquad (h \geq 0) ,$$

then we see that h^x or $h^x C_x{}^h$ is a global vector field normal to M^n, which is called the *mean curvature vector* of the submanifold M^n, and that h is a global function, which is called the *mean curvature* of the submanifold M^n. When $h_{cb}{}^x$ vanish identically, the submanifold M^n is said to be *totally geodesic*. When

$$h_{cb}{}^x = h g_{cb} C^x \qquad (h \neq 0) ,$$

$C^x = \dfrac{1}{h} h^x$ or $C^h = C^x C_x{}^h$ being a global vector field normal to M^n, M^n is said to be *totally umbilical*.

Denoting by $\Gamma_c{}^x{}_y$ the components of the connection ∇^* induced on the normal bundle $\mathcal{N}(M^n)$ from the Riemannian connection of the ambient manifold M^m, we have, by definition,

$$\Gamma_c{}^x{}_y = (\partial_c C_y{}^h + \{_j{}^h{}_i\} B_c{}^j C_y{}^i) C^x{}_h ,$$

where $C^x{}_h = C_y{}^i g^{yx} g_{ih}$. If we put

$$\nabla_c C_y{}^h = \partial_c C_y{}^h + \{_j{}^h{}_i\} B_c{}^j C_y{}^i - \Gamma_c{}^x{}_y C_x{}^h , \tag{1.9}$$

which is the *van der Waerden-Bortolotti covariant derivative* of $C_y{}^h$, then we see that $\nabla_c C_y{}^h$ is, for any fixed indices c and y, tangent to M^n. For tensor fields associated with the normal bundle $\mathcal{N}(M^n)$, ∇_c is the operator of covariant differentiation with respect to $\Gamma_c{}^x{}_y$. We thus have $\nabla_c g_{yx} = 0$, $\nabla_c g^{yx} = 0$. The van der Waerden-Bortolotti covariant differentiation ∇_c is extended to tensor fields, say $T_b{}^a{}_y{}^x$, of the mixed type on M^n in such a way that

$$\begin{aligned} \nabla_c T_b{}^a{}_y{}^x = \partial_c T_b{}^a{}_y{}^x &+ \{_c{}^a{}_e\} T_b{}^e{}_y{}^x - \{_c{}^e{}_b\} T_e{}^a{}_y{}^x \\ &+ \Gamma_c{}^x{}_z T_b{}^a{}_y{}^z - \Gamma_c{}^z{}_y T_b{}^a{}_z{}^x . \end{aligned} \tag{1.10}$$

For tensor fields, say $T_{by}{}^h$, of the mixed type on M^n, by definition we have

$$\nabla_c T_{by}{}^h = \partial_c T_{by}{}^h + \{{}_j{}^h{}_i\} B_c{}^j T_{by}{}^i - \{{}_c{}^a{}_b\} T_{ay}{}^h - \Gamma_c{}^x{}_y T_{bx}{}^h ,$$

and hence

$$\begin{aligned}\nabla_c \nabla_b C_y{}^h = \partial_c(\nabla_b C_y{}^h) + \{{}_j{}^h{}_i\} B_c{}^j \nabla_b C_y{}^i \\ - \{{}_c{}^a{}_b\} \nabla_a C_y{}^h - \Gamma_c{}^x{}_y \nabla_b C_x{}^h .\end{aligned} \tag{1.11}$$

Differentiating covariantly $G_{ji} B_b{}^j C_y{}^i = 0$, we have $G_{ji}(\nabla_c B_b{}^j) C_y{}^i + G_{ji} B_b{}^j (\nabla_c C_y{}^i) = 0$ and hence, from (1.7),

$$\nabla_c C_y{}^h = -h_c{}^a{}_y B_a{}^h , \tag{1.12}$$

since $\nabla_c C_y{}^h$ is, for any fixed indices c and y, tangent to M^n, where we have put

$$h_c{}^a{}_y = h_{ce}{}^x g^{ea} g_{xy} .$$

We use the following notations in the sequel:

$$h_{cby} = h_{cb}{}^x g_{xy} , \qquad h^{ba}{}_y = h_{dc}{}^x g^{db} g^{ca} g_{xy} , \qquad h^{bax} = h_{dc}{}^x g^{db} g^{ca} .$$

Equations (1.12) are the *equations of Weingarten* for the submanifold M^n.

We have, from (1.4) and (1.6), the Ricci formula

$$\nabla_d \nabla_c B_b{}^h - \nabla_c \nabla_d B_b{}^h = R_{kji}{}^h B_{dcb}^{kji} - K_{dcb}{}^a B_a{}^h , \tag{1.13}$$

and, from (1.9) and (1.11), the Ricci formula

$$\nabla_d \nabla_c C_y{}^h - \nabla_c \nabla_d C_y{}^h = R_{kji}{}^h B_{dc}^{kj} C_y{}^i - K_{dcy}{}^x C_x{}^h . \tag{1.14}$$

Here and in the sequel

$$B_{dcba}^{kjih} = B_d{}^k B_c{}^j B_b{}^i B_a{}^h , \qquad B_{dcb}^{kji} = B_d{}^k B_c{}^j B_b{}^i , \qquad B_{dc}^{kj} = B_d{}^k B_c{}^j$$

and $R_{kji}{}^h$, $K_{dcb}{}^a$ and $K_{dcy}{}^x$ are respectively the curvature tensors of the Riemannian metrics G of M^m, g of M^n and the induced connection ∇^* of the normal bundle $\mathcal{N}(M^n)$, the curvature tensor $K_{dcy}{}^x$ of ∇^* being defined by

$$K_{dcy}{}^x = \partial_d \Gamma_c{}^x{}_y - \partial_c \Gamma_d{}^x{}_y + \Gamma_d{}^x{}_z \Gamma_c{}^z{}_y - \Gamma_c{}^x{}_z \Gamma_d{}^z{}_y .$$

For tensor fields, say $T_b{}^a{}_y{}^x$, of the mixed type on M^n, we have, from (1.10), the Ricci formula

$$\begin{aligned}\nabla_d \nabla_c T_b{}^a{}_y{}^x - \nabla_c \nabla_d T_b{}^a{}_y{}^x \\ = K_{dce}{}^a T_b{}^e{}_y{}^x - K_{dcb}{}^e T_e{}^a{}_y{}^x + K_{dcz}{}^x T_b{}^a{}_y{}^z - K_{dcy}{}^z T_b{}^a{}_z{}^x .\end{aligned} \tag{1.15}$$

Substitution of (1.7) in the Ricci formula (1.13) gives

$$\begin{aligned}R_{kji}{}^h B_{dcb}^{kji} = K_{dcb}{}^a B_a{}^h - (h_d{}^a{}_x h_{cb}{}^x - h_c{}^a{}_x h_{db}{}^x) B_a{}^h \\ + (\nabla_d h_{cb}{}^x - \nabla_c h_{db}{}^x) C_x{}^h ,\end{aligned} \tag{1.16}$$

and substitution of (1.12) in the Ricci formula (1.14) gives

$$R_{kji}{}^h B^{kj}_{dc} C_y{}^i = K_{dcy}{}^x C_x{}^h - (h_{de}{}^x h_c{}^e{}_y - h_{ce}{}^x h_d{}^e{}_y) C_x{}^h - (\nabla_d h_c{}^a{}_y - \nabla_c h_d{}^a{}_y) B_a{}^h , \tag{1.17}$$

where $\nabla_d h_{cb}{}^x$ and $\nabla_d h_c{}^a{}_y$ are defined in the sense of (1.10), i.e.,

$$\begin{aligned} \nabla_d h_{cb}{}^x &= \partial_d h_{cb}{}^x - \{{}_d{}^e{}_c\} h_{eb}{}^x - \{{}_d{}^e{}_b\} h_{ce}{}^x + \Gamma_d{}^x{}_y h_{cb}{}^y , \\ \nabla_d h_c{}^a{}_y &= (\nabla_d h_{cb}{}^x) g^{ba} g_{xy} . \end{aligned} \tag{1.18}$$

We now have, from (1.16) and (1.17),

$$\begin{aligned} R_{kjih} B^{kjih}_{dcba} &= K_{dcba} - (h_{dax} h_{cb}{}^x - h_{cax} h_{db}{}^x) , \\ R_{kji}{}^h B^{kji}_{dcb} C^x{}_h &= \nabla_d h_{cb}{}^x - \nabla_c h_{db}{}^x , \\ R_{kji}{}^h B^{kj}_{dc} C_y{}^i C^x{}_h &= K_{dcy}{}^x - (h_{de}{}^x h_c{}^e{}_y - h_{ce}{}^x h_d{}^e{}_y) , \end{aligned} \tag{1.19}$$

where

$$R_{kjih} = R_{kji}{}^l g_{lh} , \qquad K_{dcba} = K_{dcb}{}^e g_{ea} .$$

The first, second and third equations of (1.19) are the *equations of Gauss, Codazzi* and *Ricci* respectively. Equations (1.19) altogether are sometimes called the *structure equations* of the submanifold M^n.

We now assume that the ambient manifold M^m is a space of constant curvature c, i.e., that

$$R_{kjih} = c(G_{kh} G_{ji} - G_{jh} G_{ki}) . \tag{1.20}$$

Then, substituting (1.20) in (1.19), we find

$$K_{dcba} = c(g_{da} g_{cb} - g_{ca} g_{db}) + (h_{da}{}^x h_{cbx} - h_{ca}{}^x h_{dbx}) , \tag{1.21}$$

$$0 = \nabla_d h_{cb}{}^x - \nabla_c h_{db}{}^x , \tag{1.22}$$

$$K_{dcy}{}^x = h_{de}{}^x h_c{}^e{}_y - h_{ce}{}^x h_d{}^e{}_y , \tag{1.23}$$

which are the structure equations for the submanifold M^n immersed in a space of constant curvature c. Transvection of (1.21) with g^{da} yields

$$K_{cb} = c(n-1) g_{cb} + n h^x h_{cbx} - h_{ce}{}^x h_b{}^e{}_x , \tag{1.24}$$

where $K_{cb} = K_{ecb}{}^e$ is the Ricci tensor of M^n.

When the ambient manifold M^m is a space of constant curvature c, we compute the Laplacian ΔF of a function $F = h_{cb}{}^x h^{cb}{}_x$, which is globally defined in M^n, where $\Delta = g^{cb} \nabla_c \nabla_b$. We thus have

$$\tfrac{1}{2} \Delta F = g^{ed} (\nabla_e \nabla_d h_{cb}{}^x) h^{cb}{}_x + (\nabla_c h_{ba}{}^x)(\nabla^c h^{ba}{}_x) ,$$

∇^c being defined by $\nabla^c = g^{cb} \nabla_b$.

By using the Ricci identity (1.15) and equations (1.22) of Codazzi, we find

$$\begin{aligned}\tfrac{1}{2}\Delta F &= g^{ed}[\nabla_c\nabla_e h_{bd}{}^x - K_{ecb}{}^a h_{ad}{}^x - K_{ecd}{}^a h_{ba}{}^x \\ &\qquad + K_{ecy}{}^x h_{bd}{}^y]h^{cb}{}_x + (\nabla_c h_{ba}{}^x)(\nabla^c h^{ba}{}_x) \\ &= n(\nabla_c\nabla_b h^x)h^{cb}{}_x + K_c{}^a h_{ba}{}^x h^{cb}{}_x - K_{ecba}h^{ea}{}_x h^{cbx} \\ &\qquad + K_{ecy}{}^x h_b{}^{ey} h^{cb}{}_x + (\nabla_c h_{ba}{}^x)(\nabla^c h^{ba}{}_x) ,\end{aligned}$$

where $K_c{}^a$ is defined by $K_c{}^a = K_{cb}g^{ba}$, and we have used (1.8) and equations (1.11) of Codazzi. If we substitute (1.21), (1.23) and (1.24) for $K_{ecba}, K_{ecy}{}^x$ and $K_c{}^a = K_{cb}g^{ba}$ respectively in the above equation, then we have

$$\begin{aligned}\tfrac{1}{2}\Delta F = n(\nabla_c\nabla_b h^x)h^{cb}{}_x &+ [c(n-1)g_{ca} + nh^y h_{cay} - h_{ce}{}^y h_a{}^e{}_y]h_b{}^a{}_x h^{cbx} \\ &- [c(g_{ea}g_{cb} - g_{ca}g_{eb}) + (h_{ea}{}^y h_{cby} - h_{ca}{}^y h_{eby})]h^{ea}{}_x h^{cbx} \\ &+ [h_e{}^a{}_y h_{ca}{}^x - h_c{}^a{}_y h_{ea}{}^x]h_b{}^{ey} h^{cb}{}_x + (\nabla_c h_{ba}{}^x)(\nabla^c h^{ba}{}_x) ,\end{aligned}$$

and therefore

$$\begin{aligned}\tfrac{1}{2}\Delta F = n(\nabla_c\nabla_b h^x)h^{cb}{}_x &+ cnh_{ba}{}^x h^{ba}{}_x - cn^2 h^x h_x + nh^y h_{cay} h_b{}^a{}_x h^{cbx} \\ &- h_{ea}{}^y h_{cby} h^{ea}{}_x h^{cbx} + (\nabla_c h_{ba}{}^x)(\nabla^c h^{ba}{}_x) ,\end{aligned} \tag{1.25}$$

when the ambient manifold M^m is a space of constant curvature c.

To establish some formulas for a submanifold immersed in a hypersurface for the later use, we consider an n-dimensional submanifold M^n immersed in a hypersurface M^m which is further immersed in an $(m+1)$-dimensional Riemannian manifold M^{m+1} with the metric tensor $\tilde{G}$ whose components are $\tilde{G}_{CB}$ with respect to local coordinates ζ^A.

Suppose that the local expression of M^m in M^{m+1} is

$$\zeta^A = \zeta^A(\xi^h) ,$$

where $\{\xi^h\}$ are local coordinates of M^m, and that the local expression of M^n in M^m is

$$\xi^h = \xi^h(\eta^a) ,$$

where $\{\eta^a\}$ are local coordinates of M^n. (The indices A, B, C run over the range $\{1, \cdots\cdot m+1\}$, the indices h, i, j over the range $\{1, \cdots, m\}$ and the indices a, b, c over the range $\{1, \cdots, n\}$. The summation convention is used with respect to these systems of indices.) Then the local expression of M^n in M^{m+1} is

$$\zeta^A = \zeta^A(\xi^h(\eta^a)) .$$

If we put

$$B_b{}^h = \partial_b \xi^h(\eta^a) , \qquad B_b{}^A = \partial_b \zeta^A(\xi^h(\eta^a))$$

along M^n and $B_i{}^A = \partial_i \zeta^A(\xi^h)$ along M^m, where $\partial_b = \partial/\partial\eta^b$ and $\partial_i = \partial/\partial\xi^i$, then we find $B_b{}^A = B_b{}^i B_i{}^A$ along M^n. Denote by $C_x{}^h$ $m-n$ mutually orthogonal local unit vector fields normal to M^n in M^m, and by D^A a local unit vector field normal to M^m in M^{m+1}. (The indices x, y, z run over the range $\{n+1, \cdots, m\}$. The summation convention is used with respect to this system of indices.) If we put $C_x{}^A = C_x{}^i B_i{}^A$, then $C_x{}^A$ and D^A are mutually orthogonal unit vector fields normal to M^n in M^{m+1}.

If we denote by G the metric tensor on M^m induced from the metric tensor $\tilde{G}$ of M^{m+1}, then we have, for the components of G, $G_{ji} = \tilde{G}_{CB} B_j{}^C B_i{}^B$. The contravariant components of G are denoted by G^{ji}. If we denote by g the metric tensor on M^n induced from the metric tensor G of M^m, then we have, for the components of g,

$$g_{cb} = G_{ji} B_c{}^j B_b{}^i = \tilde{G}_{CB} B_c{}^C B_b{}^B .$$

The contravariant components of g are denoted by g^{cb}.

If we denote by ∇_c and ∇_j the operators of van der Waerden-Bortolloti covariant differentiation respectively along M^n immersed in M^{m+1} and along M^m immersed in M^{m+1}, then we have

$$\nabla_c = B_c{}^j \nabla_j$$

along M^n. We now have the equations of Gauss

$$\nabla_c B_b{}^h = h_{cb}{}^x C_x{}^h , \tag{1.26}$$

$$\nabla_c B_b{}^A = H_{cb}{}^x C_x{}^A + H_{cb} D^A \tag{1.27}$$

for M^n relative to M^m and M^{m+1} respectively, where $h_{cb}{}^x$ are the second fundamental tensors of M^n relative to M^m with respect to the normals $C_x{}^h$, and $H_{cb}{}^x$ and H_{cb} are the second fundamental tensors of M^n relative to M^{m+1} with respect to the normals $C_x{}^A$ and D^A respectively. Next,

$$\nabla_j B_i{}^A = k_{ji} D^A \tag{1.28}$$

are the equations of Gauss for M^m relative to M^{m+1}, k_{ji} being the second fundamental tensor of M^m relative to M^{m+1} with respect to the normal D^A, and

$$\nabla_j D^A = -k_j{}^i B_i{}^A \tag{1.29}$$

are the equations of Weingarten for M^m relative to M^{m+1}, where $k_j{}^i = k_{jh} G^{hi}$.

Differentiating covariantly $B_b{}^A = B_b{}^i B_i{}^A$ along M^n, we obtain

$$\nabla_c B_b{}^A = (\nabla_c B_b{}^i) B_i{}^A + B_c{}^j B_b{}^i (\nabla_j B_i{}^A)$$

and hence, by substituting (1.26), (1.27) and (1.28), (1.30) $$H_{cb}{}^x C_x{}^A + H_{cb} D^A = h_{cb}{}^x C_x{}^A + B_c{}^j B_b{}^i k_{ji} D^A$$

along M^n, from which follow

(1.31) $$H_{cb}{}^x = h_{cb}{}^x , \qquad H_{cb} = B_c{}^j B_b{}^i k_{ji}$$

along M^n. If we put

(1.32) $$h^x = g^{cb} h_{cb}{}^x / n , \quad H^x = g^{cb} H_{cb}{}^x / n , \quad H = g^{cb} H_{cb} / n$$

along M^n and

(1.33) $$k = G^{ji} k_{ji} / m$$

along M^m, then we obtain, from (1.31),

(1.34) $$H^x = h^x , \qquad nH = mk - g^{yx} C_y{}^j C_x{}^i k_{ji}$$

along M^n, where h^x or $h^x C_x{}^h$ is the mean curvature vector of M^n in the normal bundle $\mathcal{N}(M^n)$ of M^n in M^m, H^x and H determine the mean curvature vector $H^x C_x{}^A + HD^A$ of M^n in the normal bundle $\mathcal{N}(M^n)$ of M^n relative to M^{m+1}, and kD^A is the mean curvature vector of M^m in M^{m+1}, g^{yx} being the contravariant components of the induced metric g^* of the normal bundle $\mathcal{N}(M^n)$ relative to the frame $\{C_x{}^h\}$.

When M^m is a totally umbilical hypersurface in a space M^{m+1} of constant curvature, we have

(1.35) $$k_{ji} = kG_{ji} ,$$

and, from (1.29) and (1.35),

(1.36) $$\nabla_j D^A = -kB_j{}^A ,$$

where the mean curvature k of M^m is determined up to a sign and is locally constant.

Next from (1.34) and (1.35) follow

(1.37) $$H^x = h^x , \qquad H = k ,$$

and thus by taking account of (1.36) we obtain

$$\begin{aligned}\nabla_c(H^x C_x{}^A + HD^A) &= \nabla_c(h^x C_x{}^A + kD^A)\\ &= (\nabla_c h^x) C_x{}^A + h^x(\nabla_c C_x{}^A) + k(\nabla_c D^A) ,\end{aligned}$$

which, together with the equations of Weingarten

$$\nabla_c C_x{}^A = -H_c{}^a{}_x B_a{}^A = -h_c{}^a{}_x B_a{}^A , \qquad \nabla_c D^A = -kB_c{}^A$$

for M^n relative to M^{m+1}, $H_c{}^a{}_x$ being defined by $H_c{}^a{}_x = H_{cb}{}^y g^{ba} g_{yx}$, implies

$$(1.38) \qquad \nabla_c(H^x C_x{}^A + HD^A) = (\nabla_c h^x) C_x{}^A - h_c{}^a{}_x h^x B_a{}^A - k^2 B_c{}^A .$$

By putting $H^{cb}{}_x = H_{ed}{}^y g^{ec} g^{db} g_{yx}$ and $H^{cb} = H_{ed} g^{ec} g^{db}$, and using (1.31) with $k_{jk} = kG_{ji}$ we thus have

$$(1.39) \qquad H_{cb}{}^x H^{cb}{}_x + H_{cb} H^{cb} = h_{cb}{}^x h^{cb}{}_x + nk^2 .$$

From (1.38) and (1.39) we hence arrive at

Lemma 1.1. *Let M^n be an n-dimensional submanifold immersed in a totally umbilical hypersurface M^m of a space M^{m+1} of constant curvature. Then the mean curvature vector $H^x C_x{}^A + HD^A$ of M^n relative to M^{m+1} is parallel in the normal bundle $'\mathcal{N}(M^n)$ of M^n in M^{m+1} if and only if the mean curvature vector $h^x C_x{}^h$ of M^n relative to M^m is parallel in the normal bundle $\mathcal{N}(M^n)$ of M^n in M^m, and the function $'F = H_{cb}{}^x H^{cb}{}_x + H_{cb} H^{cb}$ is constant in M^n if and only if the function $F = h_{cb}{}^x h^{cb}{}_x$ is constant in M^n.*

We can also prove the following lemma:

Lemma 1.2. *For a submanifold M^n in Lemma* 1.1, *the normal bundle $'\mathcal{N}(M^n)$ of M^n in M^{m+1} is locally parallelizable if the normal bundle $\mathcal{N}(M^n)$ of M^n in M^m is so also, i.e., if $R_{dcy}{}^x = 0$ in $\mathcal{N}(M^n)$.*

2. Lemmas

In this section, for later use we establish some lemmas concerning submanifolds immersed in a space of constant curvature. From (1.23) we first have

Lemma 2.1. *Let M^n be a submanifold immersed in a space of constant curvature. Then the normal bundle $\mathcal{N}(M^n)$ of M^n is locally parallelizable, i.e., $K_{dcy}{}^x = 0$, if and only if $h_b{}^{ax}$ and $h_b{}^{ay}$ are, for any indices x and y, commutative, i.e., if and only if $h_e{}^{ax} h_b{}^{ey} = h_e{}^{ay} h_b{}^{ex}$.*

From Lemma 2.1 it follows that, when $K_{dcy}{}^x = 0$, there exist certain n mutually orthogonal unit vectors $e_1{}^a, \cdots, e_n{}^a$ such that

$$h_b{}^{ax} e_\alpha{}^b = \lambda_\alpha{}^x e_\alpha{}^a \qquad (x = n+1, \cdots, m;\ \alpha \text{ not summed})$$

at each point of M^n immersed in a space of constant curvature. We call such a vector e_α with components $e_\alpha{}^a$ an eigenvector of $h_b{}^{ax}$'s, and $\lambda_\alpha{}^x$ the eigenvalue of $h_b{}^{ax}$ corresponding to e_α ($\alpha = 1, \cdots, n$). (The indices α, β, γ run over the range $\{1, \cdots, n\}$.) We shall now prove

Lemma 2.2. *Let M^n be a submanifold immersed in a space of constant curvature c, and the normal bundle $\mathcal{N}(M^n)$ be locally parallelizable. Then at each point of M^n*

$$\tfrac{1}{2}\Delta F = n(\nabla_c \nabla_b h^x)h^{cb}{}_x + (\nabla_c h_{ba}{}^x)(\nabla^c h^{ba}{}_x)$$
$$+ \sum_{\alpha<\beta} \left\{ \sum_x (\lambda_\beta{}^x - \lambda_\alpha{}^x)^2 \left(c + \sum_y \lambda_\beta{}^y \lambda_\alpha{}^y \right) \right\} .$$

Proof. We first have

$$
\begin{aligned}
&cnh_{ba}{}^x h^{ba}{}_x - cn^2 h^x h_x \\
&= c\left\{ n \sum_x \sum_\alpha (\lambda_\alpha{}^x)^2 - \sum_x \left(\sum_\alpha \lambda_\alpha{}^x \right)^2 \right\} \\
&= c\left\{ n \sum_x \sum_\alpha (\lambda_\alpha{}^x)^2 - 2 \sum_x \sum_{\alpha<\beta} \lambda_\alpha{}^x \lambda_\beta{}^x - \sum_x \sum_\alpha (\lambda_\alpha{}^x)^2 \right\} \\
&= c \sum_x \left[\sum_{\alpha<\beta} \{ (\lambda_\alpha{}^x)^2 - 2\lambda_\alpha{}^x \lambda_\beta{}^x + (\lambda_\beta{}^x)^2 \} \right] = c \sum_x \sum_{\alpha<\beta} (\lambda_\beta{}^x - \lambda_\alpha{}^x)^2 .
\end{aligned} \tag{2.1}
$$

Next,

$$
\begin{aligned}
&nh^y h_{eay} h_b{}^a{}_x h^{cbx} - h_{ea}{}^y h_{cby} h^{ea}{}_x h^{cbx} \\
&= \sum_{\alpha,\beta} \sum_{x,y} \lambda_\alpha{}^y \lambda_\beta{}^y (\lambda_\beta{}^x)^2 - \sum_{\alpha,\beta} \sum_{x,y} \lambda_\alpha{}^x \lambda_\beta{}^x \lambda_\alpha{}^y \lambda_\beta{}^y \\
&= \sum_x \sum_\alpha (\lambda_\alpha{}^x)^4 + 2 \sum_{x<y} \sum_\alpha (\lambda_\alpha{}^x)^2 (\lambda_\alpha{}^y)^2 + \sum_x \sum_{\alpha\neq\beta} \lambda_\alpha{}^x (\lambda_\beta{}^x)^3 \\
&\quad + \sum_{x\neq y} \sum_{\alpha\neq\beta} \lambda_\alpha{}^y \lambda_\beta{}^y (\lambda_\beta{}^x)^2 - \sum_x \sum_\alpha (\lambda_\alpha{}^x)^4 - 2 \sum_{x<y} \sum_\alpha (\lambda_\alpha{}^x)^2 (\lambda_\alpha{}^y)^2 \\
&\quad - 2 \sum_x \sum_{\alpha<\beta} (\lambda_\alpha{}^x)^2 (\lambda_\beta{}^x)^2 - 2 \sum_{x\neq y} \sum_{\alpha<\beta} \lambda_\alpha{}^x \lambda_\beta{}^x \lambda_\alpha{}^y \lambda_\beta{}^y \\
&= \sum_{\alpha<\beta} \sum_x \{ \lambda_\alpha{}^x (\lambda_\beta{}^x)^3 - 2(\lambda_\alpha{}^x)^2 (\lambda_\beta{}^x)^2 + \lambda_\beta{}^x (\lambda_\alpha{}^x)^3 \} \\
&\quad + \sum_{\alpha<\beta} \sum_{x\neq y} \{ \lambda_\alpha{}^y \lambda_\beta{}^y (\lambda_\beta{}^x)^2 - 2\lambda_\alpha{}^x \lambda_\beta{}^x \lambda_\alpha{}^y \lambda_\beta{}^y + \lambda_\beta{}^y \lambda_\alpha{}^y (\lambda_\alpha{}^x)^2 \} \\
&= \sum_{\alpha<\beta} \sum_x (\lambda_\beta{}^x - \lambda_\alpha{}^x)^2 \lambda_\beta{}^x \lambda_\alpha{}^x + \sum_{\alpha<\beta} \sum_{x\neq y} (\lambda_\beta{}^x - \lambda_\alpha{}^x)^2 \lambda_\beta \lambda_\alpha{}^y \\
&= \sum_{\alpha<\beta} \left\{ \sum_x (\lambda_\beta{}^x - \lambda_\alpha{}^x)^2 \sum_y \lambda_\beta{}^y \lambda_\alpha{}^y \right\} .
\end{aligned} \tag{2.2}
$$

Thus Lemma 2.2 follows from (1.25), (2.1) and (2.2).

From (1.21) we now see that the sectional curvature $\sigma_{\beta,\alpha}$ of M^n corresponding to the plane section determined by the eignevectors e_α and e_β of $h_b{}^{ax}$'s is given by

$$\sigma_{\beta,\alpha} = c + \sum_x \lambda_\beta{}^x \lambda_\alpha{}^x \qquad (\alpha \neq \beta) . \tag{2.3}$$

Thus from (1.25) and Lemma 2.2 we have

Lemma 2.3. *Under the same assumptions as in Lemma* 2.2, *we have*

$$\frac{1}{2}\Delta F = n(\nabla_c \nabla_b h^x) h^{cb}{}_x + (\nabla_c h_{ba}{}^x)(\nabla^c h^{ba}{}_x) + \sum_{\alpha<\beta} \sum_x (\lambda_\beta{}^x - \lambda_\alpha{}^x)^2 \sigma_{\beta,\alpha} \,. \tag{2.4}$$

The mean curvature vector h^x is parallel in the normal bundle $\mathscr{N}(M^n)$ if and only if

$$\nabla_c h^x = \partial_c h^x + \Gamma_c{}^x{}_y h^y = 0 \,,$$

which is equivalent to the condition that for each index c, $\tilde{\nabla}_c H^h$ is tangent to M^n, where $H^h = h^x C_x{}^h$ and $\tilde{\nabla}_c H^h = \partial_c H^h + \{{}_j{}^h{}_i\} B_c{}^j H^i$. We now proceed to establish the following Lemmas 2.4 and 2.5.

Lemma 2.4. *Let M^n be a submanifold immersed in a space of constant curvature and satisfy the conditions:*

(C) *The mean curvature vector h^x of M^n is parallel in $\mathscr{N}(M^n)$, and $\mathscr{N}(M^n)$ is locally parallelizable.*

If M^n is compact and, M^n has nonnegative sectional curvature (for all plane sections), then at every point of M^n

$$\nabla_c h_{ba}{}^x = 0 \,, \tag{2.5}$$

$$(\lambda_\beta{}^x - \lambda_\alpha{}^x)^2 \sigma_{\beta,\alpha} = 0 \qquad (\alpha \neq \beta) \tag{2.6}$$

for any indices a, b, c, α, β and x.

Proof. From (2.4), we have $\Delta F \geq 0$, since $\sigma_{\beta,\alpha} \geq 0$. Thus F is constant and therefore $\Delta F = 0$ (See, for instance, Kobayashi-Nomizu [3, Vol. I, Note 4] or Yano [6, p. 215]). Hence we have (2.5) and (2.6).

Lemma 2.5. *Let M^n be a submanifold immersed in a space of constant curvature and satisfy the condition (C) in Lemma 2.4. If $F = h_{cb}{}^x h^{cb}{}_x$ is constant on M^n, and M^n has nonnegative sectional curvature (for all plane sections), then we have the same conclusion as in Lemma 2.4.*

Now assume that M^n is a submanifold immersed in a space of constant curvature and satisfies the conditions of Lemma 2.4 or 2.5. Then, by Lemma 2.4 or 2.5, $\nabla_c h_{ba}{}^x = 0$. Since $K_{dcy}{}^x = 0$, we can choose local vector fields $C_x{}^h$ normal to M^n in such a way that $\Gamma_c{}^x{}_y = 0$, i.e., that $C_x{}^h$ are parallel in the normal bundle $\mathscr{N}(M^n)$. That is to say, for each index c, $\tilde{\nabla}_c C_x{}^h$ is tangent to M^n, where $\tilde{\nabla}_c C_x{}^h = \partial_c C_x{}^h + \{{}_j{}^h{}_i\} B_c{}^j C_x{}^i$. Assume in the sequel that if $K_{dcy}{}^x = 0$, then the normal vector fields $C_x{}^h$ are chosen in the way mentioned above. Thus, if $K_{dcy}{}^x = 0$, then by (1.18), $\nabla_c h_{ba}{}^x = 0$ reduces to

$$\nabla_c h_{ba}{}^x = \partial_c h_{ba}{}^x - \{{}_c{}^e{}_b\} h_{ea}{}^x - \{{}_c{}^e{}_a\} h_{be}{}^x = 0 \,, \tag{2.7}$$

which implies that all the eigenvalues $\lambda_\alpha{}^x$ of $h_b{}^{ax}$ are locally constant and that each eigenspace of $h_b{}^{ax}$ is of constant dimension. Let $v_1{}^a, \cdots, v_n{}^a$ be mutually orthogonal local unit vector fields in M^n, which are the eigenvectors of all $h_b{}^{ax}$

at each point, and let $\lambda_\alpha{}^x$ be the eigenvalue of $h_b{}^{ax}$ corresponding to $v_\alpha{}^a$. We call each of $v_\alpha{}^a$'s an eigenvector field of $h_b{}^{ax}$, and denote by λ_α the normal vector field with components $\lambda_\alpha{}^h = \lambda_\alpha{}^x C_x{}^h$, which is globally defined in $\mathcal{N}(M^n)$ and is called the *vector of eigenvalues* of M^n corresponding to $v_\alpha{}^a$. If, for a vector λ_α of eigenvalues, all the eigenvector fields corresponding to λ_α form a p_α-dimensional distribution, then we say that the multiplicity of λ_α is p_α. If we, for instance, fix the choice of the normals $C_x{}^h$, then we can identify λ_α with a vector of R^{m-n} having components $(\lambda_\alpha{}^{n+1}, \cdots, \lambda_\alpha{}^m)$, where the usual inner product (λ, μ) is defined in R^{m-n}. Thus, in terms of such an identification, we shall prove

Lemma 2.6. *Let M^n be a submanifold immersed in a space of constant curvature, say c, and assume that M^n satisfies the conditions of Lemma 2.4 or 2.5. Then there exists a certain number of distinct vectors $\mu_1, \cdots, \mu_N$ of R^{m-n} $(N \leq n)$, whose inner products are given by*

$$(\mu_A, \mu_B) = -c \qquad (A \neq B;\, A, B = 1, \cdots, N) , \tag{2.8}$$

in such a way that any vector of eigenvalues of M^n coincides with one of $\mu_1, \cdots, \mu_N$ and any of $\mu_1, \cdots, \mu_N$ is a vector of eigenvalues.

Proof. First, assume that all sectional curvatures of M^n vanish, i.e., that $\sigma_{\alpha,\beta} = 0$. Then, from (2.3),

$$(\lambda_\alpha, \lambda_\beta) = -c \qquad (\alpha \neq \beta) .$$

Thus λ_α's themselves have the property (2.8).

Next, assume that there exists a nonzero $\sigma_{\beta,\alpha}$. Then we may suppose that $\sigma_{1,2}, \cdots, \sigma_{1,p}$ are nonzero and $\sigma_{1,p+1} = \cdots = \sigma_{1,n} = 0$. Thus, by (2.6),

$$\lambda_1 = \cdots = \lambda_p = \mu_1 ,$$

and, by (2.3),

$$(\lambda_q, \lambda_1) = -c \qquad (q > p) .$$

If we now take account of (2.3), we find

$$\sigma_{\beta,\alpha} = \sigma_{1,2} \qquad (\beta < \alpha;\, \alpha, \beta = 1, \cdots, p) ,$$
$$\sigma_{\beta,q} = 0 \qquad (\beta = 1, \cdots, p;\, q = p+1, \cdots, n) .$$

If $\sigma_{p+1,\beta+2}, \cdots, \sigma_{p+1,r}$ are nonzero, and $\sigma_{p+1,r+1} = \cdots = \sigma_{p+1,n} = 0$, then

$$\lambda_{p+1} = \cdots = \lambda_r = \mu_2 ,$$

and

$$(\lambda_q, \mu_2) = -c \qquad (q > r) ,$$
$$\sigma_{\beta,\alpha} = \sigma_{p+1,p+2} \qquad (\beta < \alpha; \alpha, \beta = p+1, \cdots, r) ,$$
$$\sigma_{\beta,q} = 0 \qquad (\beta = p+1, \cdots, r; q = r+1, \cdots, n) .$$

In this way, we shall have

$$\lambda_{r+1} = \cdots = \lambda_p = \mu_3 , \quad (\lambda_q, \mu_3) = -c \quad (q > s) ;$$
$$\cdots\cdots\cdots\cdots$$

as far as there exists a non-zero $\sigma_{\beta,\alpha}$.

If $\sigma_{\beta,\alpha} = 0$ for $\beta < \alpha$ ($\alpha, \beta = t, \cdots, n$; $t > 1$), then we put

$$\lambda_t = \mu_B, \cdots, \lambda_n = \mu_N .$$

Thus from (2.3) we have

$$(\lambda_q, \mu_B) = \cdots = (\lambda_q, \mu_N) = -c \qquad (q \geq t) ,$$

so that these $\mu_1, \cdots, \mu_N$ have the properties of the lemma.

We shall now prove the following algebraic lemma for later use.

Lemma 2.7. *Let $\mu_1, \cdots, \mu_N$ be distinct vectors belonging to R^s such that*

$$(\mu_A, \mu_B) = k \qquad (A \neq B; A, B = 1, \cdots, N) .$$

If $\mu_1, \cdots, \mu_N$ span an r-dimensional subspace ($s \geq r > 0$), then $N = r$ or $N = r+1$ and hence $N \leq s+1$. Furthermore in the last case where $N = r+1$, we have

$$\begin{vmatrix} (\mu_1, \mu_1) & k & \cdots & k \\ k & (\mu_2, \mu_2) & \cdots & k \\ \cdots & \cdots & \cdots & \cdots \\ k & k & \cdots & (\mu_N, \mu_N) \end{vmatrix} = 0 , \tag{2.9}$$

and one of $\mu_1, \cdots, \mu_N$ is necessarily zero when $k = 0$.

Proof. First assume that $k \neq 0$. Then none of $\mu_1, \cdots, \mu_N$ vanishes. If $N > r+1$ and $\mu_1, \cdots, \mu_N$ span an r-dimensional subspace, then we may suppose that $\mu_1, \cdots, \mu_r$ are linearly independent. Putting

$$\mu_{r+1} = a_1\mu_1 + \cdots + a_r\mu_r ,$$

taking the inner product with μ_{r+2} and μ_1, and using $(\mu_A, \mu_B) = k$ $(A \neq B)$, we obtain respectively

$$a_1 + \cdots + a_r = 1 , \qquad a_1((\mu_1, \mu_1) - k) = 0 .$$

Thus we may assume that

$$\mu_{r+1} = a_1\mu_1 + \cdots + a_t\mu_t \qquad (t \le r) ,$$
$$(\mu_A, \mu_A) = k \qquad (A = 1, \cdots, t) ,$$

so that

$$(\mu_A, \mu_B) = k \qquad (A, B = 1, \cdots, t) ,$$

which contradicts the independence of $\mu_1, \cdots, \mu_r$, since

$$\begin{vmatrix} (\mu_1, \mu_1) & \cdots & (\mu_1, \mu_t) \\ & \cdots\cdots & \\ (\mu_t, \mu_1) & \cdots & (\mu_t, \mu_t) \end{vmatrix} = 0$$

for $t \neq 1$, and $\mu_{r+1} = \mu_1$ for $t = 1$. Thus we have $N \le r + 1$.

When $N = r + 1$, we have a nontrivial linear relation

$$a_1\mu_1 + \cdots + a_N\mu_N = 0$$

and therefore, by taking inner products with $\mu_1, \cdots, \mu_N$ in turn,

$$\begin{aligned} a_1(\mu_1, \mu_1) + a_2k \quad + \cdots + \quad a_Nk \quad &= 0 , \\ a_1k \quad + a_2(\mu_2, \mu_2) + \cdots + \quad a_Nk \quad &= 0 , \\ \cdots\cdots\cdots\cdots & \\ a_1k \quad + a_2k \quad + \cdots + a_N(\mu_N, \mu_N) &= 0 , \end{aligned}$$

respectively, which imply (2.9) because of $(a_1, \cdots, a_N) \neq (0, \cdots, 0)$. When $k = 0$, the lemma is obviously true. Thus Lemma 2.7 is proved.

Let M^n be a submanifold immersed in a space of constant curvature, and suppose that M^n satisfies the condition of Lemma 2.4 or 2.5. Then for a vector μ_α of eigenvalues all the corresponding eigenvector fields span a distribution D_α, and for a vector field v^a belonging to D_α we have

$$h_b{}^{ax}v^b = \mu_\alpha{}^x v^a . \tag{2.10}$$

Thus

$$h_b{}^{ax}\nabla_c v^b = \mu_\alpha{}^x \nabla_c v^a$$

by (2.7) and the constancy of $\mu_\alpha{}^x$, so that the distribution D_α and the orthogonal complement $\bar{D}_\alpha$ of D_α are both integrable and that the integral manifolds of D_α and $\bar{D}_\alpha$ are totally geodesic in M^n. Hence M^n is locally a pythagorean product $M_\alpha \times \bar{M}_\alpha$, where M_α and $\bar{M}_\alpha$ are respectively some integral manifolds of D_α and $\bar{D}_\alpha$. For any vector fields u^a and v^a tangent to M_α, from (2.10) we have

$$u^c\nabla_c(v^bB_b{}^h) = (u^c\nabla_c v^b)B_b{}^h + h_\alpha(g_{cb}u^cv^b)C_\alpha{}^h \qquad (\mu_\alpha \neq 0) ,$$

$$u^c\nabla_c(v^bB_b{}^h) = (u^c\nabla_c v^b)B_b{}^h \qquad (\mu_\alpha = 0) ,$$

where

$$h_\alpha = \{\sum_x (\mu_\alpha{}^x)^2\}^{1/2} , \qquad C_\alpha{}^h = \mu_\alpha{}^x C_x{}^h / h_\alpha .$$

Thus, when $\dim M_\alpha \geq 2$, the submanifold M_α is totally umbilical or totally geodesic in M^m according as the mean curvature vector μ_α of M_α is nonzero or zero.

When $\dim M_\alpha = 1$ and $\mu_\alpha \neq 0$, M_α is a curve in M^m whose first curvature along M_α is constant. For simplicity such a curve is called a *totally umbilical submanifold of dimension* 1 in M^m. When $\dim M_\alpha = 1$ and $\mu_\alpha = 0$, M_α is a geodesic arc of M^m, which is, for simplicity, called a *totally geodesic submanifold of dimension* 1 in M^m. Thus we have

Lemma 2.8. *Let M^n be a submanifold immersed in a space M^m of constant curvature, and assume that M^n satisfies the condition of Lemma* 2.4 *or* 2.5. *If distinct vectors of eigenvalues of M^n are given by $\mu_1, \cdots, \mu_N$, then M^n is locally a phthagorean product $M_1 \times \cdots \times M_N$, where M_α is a totally umbilical or totally geodesic submanifold in M^m according as the mean curvature vector μ_α $(\alpha = 1, \cdots, N)$ of M_α is nonzero or zero.*

Let M^n be a submanifold immersed in an m-dimensional Euclidean space R^m, and denote by N_P the normal space of M^n at a point P of M^n. The subspace $'N_P(\subset N_P)$ spanned by normal vectors $v^cu^bh_{cb}{}^xC_x{}^h$, u^a and v^a being arbitrary tangent vectors of M^n at P, is assumed to be of constant dimension r, i.e., $\dim {}'N_P = r$ is independent of P $(1 \leq r < m - n)$. Thus $'\mathcal{N}(M^n) = \bigcup_{P \in M^n} {}'N_P$ is a subbundle of the normal bundle $\mathcal{N}(M^n)$. Take mutually orthogonal r local unit vector fields $C_A{}^h$ in $'\mathcal{N}(M^n)$ and mutually orthogonal $m - n - r$ local unit vector fields $C_p{}^h$, which are normal to M^n and $C_A{}^h$. (The indices A, B, C run over the range $\{n + 1, \cdots, n + r\}$ and the indices p, q, r over the range $\{n + r + 1, \cdots, m\}$. The summation convention is used with respect to the system of indices A, B, C.) Then equations (1.7) of Gauss and equations (1.12) of Weingarten for the submanifold M^n reduce respectively to

$$\nabla_c B_b{}^h = h_{cb}{}^B C_B{}^h , \qquad h_{cb}{}^P = 0 , \tag{2.11}$$

and

$$\nabla_c C_B{}^h = -h_c{}^a{}_B B_a{}^h , \tag{2.12}$$

$$\nabla_b C_q{}^h = 0 . \tag{2.13}$$

Next, from the structure equation (1.23) for the submanifold M^n, we have

(2.14) $$K_{dcq}{}^p = 0 ,$$

which shows that the vector bundle $'\mathcal{N}(M^n)$ is locally parallelizable. Thus we can choose $C_q{}^h$ in $'\mathcal{N}(M^n)$ in such a way that

(2.15) $$\nabla_c C_q{}^h = \partial_c C_q{}^h + \{_j{}^h{}_i\} B_c{}^j C_q{}^i .$$

If we assume that (ξ^h) is a system of rectangular coordinates in R^m, then from (2.15) we obtain

$$\nabla_c C_q{}^h = \partial_c C_q{}^h ,$$

from which (2.13) it follows that all the components $C_q{}^h$ are constant. On the other hand, since $B_c{}^h$ and $C_q{}^h$ are mutually orthogonal, we have

$$\sum_{h=1}^{m} C_q{}^h B_c{}^h = 0 , \qquad B_c{}^h = \partial\xi^h/\partial\eta^c ,$$

which gives, by integration,

$$\sum_{h=1}^{m} C_q{}^h \xi^h(\eta^a) = D_q ,$$

where D_q are constant and $\xi^h = \xi^h(\eta^a)$ is the local expression of M^n in R^m. Thus the submanifold M^n lies in an $(n + r)$-dimensional plane, defined by the equations $\sum_{h=1}^{m} C_q{}^h \xi^h = D_q$, of the ambient Euclidean space R^m. Consequently, we obtain

Lemma 2.9. *For a submanifold M^n immersed in an m-dimensional Euclidean space R^m, if the normal space $'N_P$ spanned by $v^c u^b h_{cb}{}^x C_x{}^h$, u^a and v^a being arbitrary vectors tangent to M^n at $P \in M^n$, is of constant dimension $r\,(1 \leq r < m - n)$, i.e., if r is independent of P, then M^n is immersed in an $(n + r)$-dimensional plane of R^m.*

By similar arguments as above, we have

Lemma 2.10. *For a submanifold M^n immersed in an m-dimensional sphere S^m defined by an equation $(x, x) = a^2$ $(a > 0)$ in an $(m + 1)$-dimensional Euclidean space R^{m+1} with usual inner product (x, y), if the normal space $'N_p$ (appearing in Lemma 2.9) is of constant dimension r $(1 \leq r < m - n)$, then M^n is immersed in a great sphere S^{n+r} of S^m defined by equations $(x, x) = a^2$ $(a > 0)$, $(x, e_1) = 0, \cdots, (x, e_{m-n-r}) = 0$, $e_1, \cdots, e_{m-n-r}$ being linearly independent unit vectors.*

If M^n is a submanifold immersed in an m-dimensional Euclidean space R^m (or in an m-dimensional sphere S^m) and statisfies the conditions of Lemma 2.4 (or 2.5), then the vectors of eigenvalues of the submanifold M^n span the subspace $'N_P$ appearing in Lemmas 2.9 and 2.10. Thus from Lemmas 2.9 and 2.10 we obtain

Lemma 2.11. *Let M^n be a submanifold immersed in an m-dimensional Euclidean space R^m (resp. sphere S^m) and satisfy the conditions of Lemma 2.4 or 2.5. If the vectors of eigenvalues of M^n span an r-dimensional $(0 \leq r < m-n)$ subspace in the normal space to M^n at each point of M^n, then M^n is immersed in an $(n+r)$-dimensional plane in R^m (resp. great sphere in S^m) and there exists in R^m (resp. S^m) no plane (resp. great sphere) of dimension less than $n+r$ which contains M^n $(1 \leq r < m-n)$.*

A submanifold M^n immersed in an m-dimensional Euclidean space R^m (resp. sphere S^m) is said to be of *essential codimension* r $(0 \leq r < m-n)$, if there exists in R^m (resp. S^m) an $(n+r)$-dimensional plane $\bar{R}^{n+r}$ (resp. great sphere $\bar{S}^{n+r}$) containing M^n and no such a plane (resp. great sphere) of dimension less than $n+r$. A submanifold M^n immersed in R^m (resp. S^m) is said to be of *essential codimension* $m-n$, if there exists in R^m (resp. S^m) no plane (resp. great sphere) containing M^n.

3. Submanifolds in a Euclidean space

We first explain a few examples of n-dimensional submanifolds in an m-dimensional Euclidean space R^m with usual inner product (x, y). For integers $p_1, \cdots, p_N$ such that $p_1, \cdots, p_N \geq 1$, $p_1 + \cdots + p_N = n$, consider R^m as $R^{p_1+1} \times \cdots \times R^{p_N+1}$, where $N = m-n$, and let

$$S^{p_1}(r_1) = \{x_1 \in R^{p_1+1}, (x_1, x_1) = r_1^2\}\,,$$
$$\cdots\cdots\cdots$$
$$S^{p_N}(r_N) = \{x_N \in R^{p_N+1}, (x_N, x_N) = r_N^2\}\,.$$

Then the pythagorean product

$$S^{p_1}(r_1) \times \cdots \times S^{p_N}(r_N) = \{(x_1, \cdots, x_N) \in R^m, x_\alpha \in S^{p_\alpha}(r_\alpha), \alpha = 1, \cdots, N\}$$

is an n-dimensional submanifold M^n of essential codimension $m-n$ in R^m and its vectors of eigenvalues are given by

$$\mu_1 = r_1^{-2}x_1, \cdots, \mu_N = r_N^{-2}x_N \tag{3.1}$$

at $(x_1, \cdots, x_N) \in M^n$, whose multiplicities are $p_1, \cdots, p_N$ respectively. Thus the mean curvature vector field H of M^n is given by

$$H = (p_1\mu_1 + \cdots + p_N\mu_N)/n = (p_1 r_1^{-2}x_1 + \cdots + p_N r_N^{-2}x_N)/n \tag{3.2}$$

at $(x_1, \cdots, x_N) \in M^n$, which is parallel in the normal bundle $\mathcal{N}(M^n)$ of M^n, and the function $F = h_{cb}{}^x h^{cb}{}_x$ is given by

$$F = (\mu_1, \mu_1) + \cdots + (\mu_N, \mu_N) = 1/r_1^2 + \cdots + 1/r_N^2 , \tag{3.3}$$

which is constant in M^n. It is easily verified that the normal bundle $\mathcal{N}(M^n)$ is locally parallelizable.

For integers $p_1, \cdots, p_N, p$ such that $p_1, \cdots, p_N, p \geq 1, p_1 + \cdots + p_N + p = n$, consider R^m as $R^{p_1+1} \times \cdots \times R^{p_N+1} \times R^p$, where $N = m - n$. Then the pythagorean product

$$\begin{aligned} &S^{p_1}(r_1) \times \cdots \times S^{p_N}(r_N) \times R^p \\ &\quad = \{(x_1, \cdots, x_N, x) \in R^m, x_\alpha \in S^{p_\alpha}(r_\alpha), \alpha = 1, \cdots, N, x \in R^p\} \end{aligned}$$

is an n-dimensional submanifold M^n of essential codimension $N = m - n$ in R^m. The vectors $\mu_1, \cdots, \mu_N$ of eigenvalues, the mean curvature vector H and the function F are given respectively by (3.1), (3.2) and (3.3) at $(x_1, \cdots, x_N, x) \in M^n$. Thus H is parallel in the normal bundle $\mathcal{N}(M^n)$, F is constant in M^n and $\mathcal{N}(M^n)$ is locally parallelizable.

Using the same arguments as those developed by Nomizu and Smyth (See [4, Theorem 1]), from Lemmas 2.8 and 2.10 we have

Theorem 3.1. *Let M^n be a complete submanifold of dimension n immersed in a Euclidean space R^m of dimension m $(1 < n < m)$ with nonnegative sectional curvature. Suppose that the normal bundle $\mathcal{N}(M^n)$ is locally parallelizable and that the mean curvature vector of M^n is parallel in $\mathcal{N}(M^n)$. If the function $F = h_{cb}{}^x h^{cb}{}_x$ is constant in M^n, then M^n is a sphere $S^n(r)$ of dimension n, an n-dimensional plane $R^n(\subset R^m)$, a pythagorean product of the form*

$$\begin{aligned} &S^{p_1}(r_1) \times \cdots \times S^{p_N}(r_N) , \\ &p_1, \cdots, p_N \geq 1 , \quad p_1 + \cdots + p_N = n , \quad 1 < N \leq m - n , \end{aligned} \tag{3.4}$$

or a pythagorean product of the form

$$\begin{aligned} &S^{p_1}(r_1) \times \cdots \times S^{p_N}(r_N) \times R^p , \\ &p_1, \cdots, p_N, p \geq 1 , \quad p_1 + \cdots + p_N + p = n , \quad 1 < N \leq m - n , \end{aligned} \tag{3.5}$$

where $S^p(r)$ is a p-dimensional sphere with radius r, and $R^p(\subset R^m)$ a p-dimensional plane. If M^n is a pythagorean product of the form (3.4) *or* (3.5), *then M^n is of essential codimension N.*

Finally, from Lemmas 2.8 and 2.10 we have

Theorem 3.2. *Let M^n be a compact submanifold of dimension n immersed in a Euclidean space R^m of dimension m $(1 < n < m)$ with nonnegative sectional curvature, and suppose that the normal bundle $\mathcal{N}(M^n)$ of M^n is locally parallelizable. If the mean curvature vector of M^n is parallel in $\mathcal{N}(M^n)$, then M^n is an n-dimensional sphere $S^n(r)$ or a pythagorean product of the form* (3.4), *which is of essential codimension N.*

Remark. Suppose that a submanifold M^n immersed in R^m satisfies the conditions of Theorem 3.1 or 3.2, and is of essential codimension s less than $m - n$. Then M^n is contained in a plane $\bar{R}^{m+s}$ of R^m, and satisfies the same conditions as those mentioned in Theorem 3.1 or 3.2 and satisfied by M^n considered as a submanifold in R^m if M^n is considered as a submanifold in $\bar{R}^{m+s}$.

4. Submanifolds in a sphere

In an $(m + 1)$-dimensional Euclidean space R^{m+1} with usual inner product (x, y),

$$S^m(a) = \{x \in R^{m+1}, (x, x) = a^2\}$$

is called an m-dimensional sphere of radius $a > 0$. For mutually orthogonal unit vectors $b_1, \cdots, b_{m-n}$ in R^{m+1}, a submanifold $\sum^n (r)$ defined in $S^m(a)$ by

$$\textstyle\sum^n (r) = \{x \in S^m(a), (x, b_\beta) = d_\beta, \beta = 1, \cdots, m - n\}$$

is called an n-dimensional small sphere of $S^m(a)$ with radius r if $(d_1, \cdots, d_{m-n}) \neq (0, \cdots, 0)$, where $r^2 = a^2 - d_1^2 - \cdots - d_{m-n}^2 > 0$ and $1 < n < m$. $\sum^n (r)$ is called an n-dimensional great sphere of $S^m(a)$, if $(d_1, \cdots, d_{m-n}) = (0, \cdots, 0)$, i.e., if $r = a$. If $r \neq a$, a small sphere $\sum^n (r)$ is a totally umbilical submanifold of essential codimension $m - n$ in $S^m(a)$, and the mean curvature h relative to $S^m(a)$ is given by

$$h = d/(a\sqrt{a^2 - d^2}) , \quad d^2 = d_1^2 + \cdots + d_{m-n}^2 \quad (d > 0) . \tag{4.1}$$

A great sphere $\sum^n (a)$ is totally geodesic in $S^m(a)$ and of essential codimension 0.

We explain other examples of n-dimensional submanifolds in $S^m(a)$. For integers $p_1, \cdots, p_N$ such that $p_1, \cdots, p_N \geq 1, p_1 + \cdots + p_N = n$, consider R^{m+1} as $R^{p_1+1} \times \cdots \times R^{p_N+1}$, where $N = m - n + 1$. Then

$$\begin{aligned} &S^{p_1}(r_1) \times \cdots \times S^{p_N}(r_N) \\ &\quad = \{(x_1, \cdots, x_N) \in R^{m+1}, x_\alpha \in R^{p_\alpha}(r_\alpha), \alpha = 1, \cdots, N\} , \end{aligned} \tag{4.2}$$

where $S^{p_\alpha}(r_\alpha) \subset R^{p_\alpha+1} (\alpha = 1, \cdots, N)$, is an n-dimensional submanifold M^n of essential codimension $m - n$ imbedded in $S^m(a)$ if

$$r_1^2 + \cdots + r_N^2 = a^2 . \tag{4.3}$$

Thus from (1.30) with $k_{ji} = a^{-2}G_{ji}$ it follows that the vectors of eigenvalues of M^n relative to $S^m(a)$ are given by

$$\begin{aligned} \mu_1 &= r_1^{-2}x_1 - a^{-2}(x_1 + \cdots + x_N) , \\ &\cdots\cdots\cdots\cdots \\ \mu_N &= r_N^{-2}x_N - a^{-2}(x_1 + \cdots + x_N) \end{aligned}$$

at $(x_1, \cdots, x_N) \in M^n$, whose multiplicities are respectively $p_1, \cdots, p_N$, and therefore that the mean curvature vector H of M^n relative to $S^m(a)$ is given by

$$
\begin{aligned}
H &= (p_1\mu_1 + \cdots + p_N\mu_N)/n \\
&= \frac{1}{n}(p_1 r_1^{-2} x_1 + \cdots + p_N r_N^{-2} x_N)/n - a^{-2}(x_1 + \cdots + x_N)
\end{aligned} \tag{4.4}
$$

at $(x_1, \cdots, x_N) \in M^n$, which is parallel in the normal bundle $\mathscr{N}(M^n)$ of M^n relative to $S^m(a)$, and the function $F = h_{cb}{}^x h^{cb}{}_x$ by

$$
\begin{aligned}
F &= (\mu_1, \mu_1) + \cdots + (\mu_N, \mu_N) \\
&= r_1^2(r_1^{-2} - a^{-2})^2 + \cdots + r_N^2(r_N^{-2} - a^{-2})^2 + N(N-1)a^{-2}
\end{aligned}
$$

which is constant in M^n. It is easily verified that the normal bundle $\mathscr{N}(M^n)$ is locally parallelizable.

Let $\sum^{m-1}(r)$ be an $(m-1)$-dimensional small sphere of $S^m(a)(0 < r < a)$. For integers $p_1, \cdots, p_{N'}$ such that $p_1, \cdots, p_{N'} \geq 1$, $p_1 + \cdots + p_{N'} = n$, $N' = m - n$, in $\sum^{m-1}(r)$ consider an n-dimensional submanifold $'M^n$ of the form

$$\textstyle\sum^{p_1}(r_1) \times \cdots \times \sum^{p_{N'}}(r_{N'}) \subset \sum^{m-1}(r)\,, \tag{4.5}$$

$$r_1^2 + \cdots + r_{N'}^2 = r^2 < a^2\,, \tag{4.6}$$

where $\sum^{p_\alpha}(r_\alpha)(\alpha = 1, \cdots, N')$ is a p_α-dimensional sphere with radius r_α, and $'M^n$ is constructed in $\sum^{m-1}(r)$ in the same way as that used in constructing in $S^m(a)$ a submanifold M^n of the form (4.2). Then $'M^n$ is an n-dimensional submanifold of essential codimension $m - n - 1$ in $\sum^{m-1}(r)$ and therefore $m - n$ in $S^m(a)$. The mean curvature vector of $'M^n$ relative to $S^m(a)$ is parallel in the normal bundle $\mathscr{N}('M^n)$ of $'M^n$ relative to $S^m(a)$, the function $F = h_{cb}{}^x h^{cb}{}_x$, $h_{cb}{}^x$ being the second fundamental tensors of $'M^n$ relative to $S^m(a)$, is constant in $'M^n$, and the normal bundle $\mathscr{N}('M^n)$ relative to $S^m(a)$ is locally parallelizable.

We shall now prove

Theorem 4.1. *Let M^n be a complete submanifold of dimension n immersed in an m-dimensional sphere $S^m(a)$ with radius a $(0 < a, 1 < n < m)$ and non-negative sectional curvature. Suppose that the mean curvature vector of M^n is parallel in the normal bundle $\mathscr{N}(M^n)$ and that $\mathscr{N}(M^n)$ is locally parallelizable. If the function $F = h_{cb}{}^x h^{cb}{}_x$ is constant in M^n, then M^n is a small sphere $\sum^n(r)$, a great sphere $\sum^n(a)$ or a pythagorean product of a certain number of speres. Moreover, if M^n is of essential codimension $m - n$, then M^n is a pythagorean product of the form* (4.2) *with $r_1^2 + \cdots + r_N^2 = a^2$, $N = m - n + 1$, or of the form* (4.5) *with $r_1^2 + \cdots + r_{N'}^2 = r^2 < a^2$, $N' = m - n$. If M^n is a pythagorean product of the form* (4.5) *with $r_1^2 + \cdots + r_N^2 = r^2 < a^2$, $N = m - n$, then M^n is contained in a small sphere $\sum^{m-1}(r)$ of $S^m(a)$.*

Proof. If M^n is considered as a submanifold immersed in R^{m+1}, then from Lemmas 1.1 and 1.2 the mean curvature vector of M^n relative to R^{m+1} is parallel in the normal bundle $'\mathcal{N}(M^n)$ of M^n in R^{m+1}, the function $'F = H_{cb}{}^x H^{cb}{}_x + H_{cb}H^{cb}, H_{cb}{}^x$ and H_{cb} being the second fundamental tensors of M^n relative to R^{m+1}, is constant in M^n, and $'\mathcal{N}(M^n)$ is locally parallelizable. Thus, by Theorem 3.1, M^n is an n-dimensional sphere or a pythagorean product of a certain number of spheres, since $M^n(\subset S^m(a))$ is bounded. Hence M^n is a small sphere of $S^m(a)$, a great sphere of $S^m(a)$ or a pythagorean product of a certain number of spheres.

When M^n is of essential codimension $m - n$ in $S^m(a)$, there exist $m - n$ or $m - n + 1$ distinct vectors of eigenvalues of M^n relative to $S^m(a)$ and hence $m - n$ or $m - n + 1$ distinct vectors of eigenvalues of M^n relative to R^{m+1}. Thus M^n is of essential codimension $m - n$ or $m - n + 1$ in R^{m+1}. If M^n is of essential codimension $m - n$ in R^{m+1} then it is contained in a certain m-dimensional plane $R^m(\subset R^{m+1})$(See Theorem 3.1), not passing through the origin of R^{m+1}. Otherwise M^n is not of essential codimension $m - n$ in $S^m(a)$. Thus, if R^m is of essential codimension $m - n$ in R^{m+1}, then M^n is a pythagorean product of the form (4.5) satisfying (4.6). When M^n is of essential codimension $m - n + 1$ in R^{m+1}, M^n is a pythagorean product of the form (4.2) satisfying (4.3). Hence Theorem 4.1 is proved.

By similar devices as in the proof of Theorems 3.1, 3.2 and 4.1, from Lemmas 2.8 and 2.10 we have

Theorem 4.2. *Let M^n be a compact submanifold of dimension n immersed in an m-dimensional sphere $S^m(a)(1 < n < m)$ with nonnegative sectional curvature. Suppose that the normal bundle $\mathcal{N}(M^n)$ of M^n is locally parallelizable and that the mean curvature vector of M^n is parallel in $\mathcal{N}(M^n)$. If M^n is of essential codimension $m - n$, then we have the same conclusion as in Theorem* 4.1.

Remark. If a submanifold M^n immersed in $S^m(a)$ satisfies the conditions of Theorem 4.1 or 4.2 and if M^n is of essential codimension s less than $m - n$, then M^n is contained in a great sphere S^{n+s} of S^m, and satisfies the same conditions as those mentioned in Theorem 4.1 or 4.2 and satisfied by M^n considered as a submanifold in S^m if M^n is considered as a submanifold in $\bar{S}^{n+s}$.

5. Minimal submanifolds in spheres

A submanifold is said to be *minimal* if its mean curvature vanishes identically.

Let M^n be a submanifold immersed in an m-dimensional sphere S^m and satisfy the conditions in Theorem 4.1 or 4.2. Then by (4.4) the mean curvature H of M^n is given by

$$H = (p_1\mu_1 + \cdots + p_N\mu_N)/n ,$$

where $\mu_1, \cdots, \mu_N$ are the distinct vectors of eigenvalues, and $p_1, \cdots, p_N$ the multiplicities of $\mu_1, \cdots \mu_N$ respectively. Since the mean curvature h is defined by $h^2 = g_{yx}H^yH^x$ $(h \geq 0)$, H^x being the components of H, such a submanifold M^n is minimal if and only if

$$H = p_1\mu_1 + \cdots + p_N\mu_N = 0 . \tag{5.1}$$

By using Theorem 4.1 we shall now prove

Theorem 5.1. *Let M^n be a complete minimal submanifold of dimension n immersed in an m-dimensional sphere $S^m(a)$ with radius a $(0 < a, 1 < n < m)$ and nonnegative sectional curvature, and suppose the normal bundle $\mathcal{N}(M^n)$ of M^n is locally parallelizable. If the function $F = h_{cb}{}^x h^{cb}{}_x$ is constant in M^n, then M^n is a great sphere of $S^m(a)$ or a pythagorean product of the form*

$$\begin{gathered} S^{p_1}(r_1) \times \cdots \times S^{p_N}(r_N) , \\ p_1, \cdots, p_N \geq 1 , \quad p_1 + \cdots + p_N = n , \quad 1 < N \leq m - n + 1 \end{gathered} \tag{5.2}$$

with essential codimension $N - 1$, where

$$r_\alpha = a\sqrt{p_\alpha/n} \qquad (\alpha = 1, \cdots, N) . \tag{5.3}$$

Proof. Since M^n is minimal, we see, from (5.1), that the vectors $\mu_1, \cdots, \mu_N$ of eigenvalues are linearly dependent. Thus from Lemmas 2.7 and 2.11 it follows that M^n is of essential codimension $N - 1$ if M^n is a pythagorean product of the form (5.2). We find (5.3) from (4.4). Thus Theorem 5.1 is proved.

We can prove

Theorem 5.2. *Let M^n be a compact minimal submanifold of dimension n immersed in an m-dimensional sphere $S^m(a)$ with radius a $(0 < a, 1 < n < m)$. If M^n has nonnegative sectional curvature and the normal bundle $\mathcal{N}(M^n)$ of M^n is locally parallelizable, then we have the same conclusion as in Theorem 5.1.*

We now explain a few n-dimensional minimal submanifolds M^n of essential codimension $m - n$ in an m-dimensional sphere $S^m(a)$ for small m and n as follows:

In $S^3(a)$ $\quad S^1(a/\sqrt{2}) \times S^1(a/\sqrt{2}) \quad (n = 2)$.

In $S^4(a)$ $\quad S^2(a\sqrt{2/3}) \times S^1(a/\sqrt{3}) \quad (n = 3)$.

In $S^5(a)$ $\quad S^1(a/\sqrt{3}) \times S^1(a/\sqrt{3}) \times S^1(a/\sqrt{3}) \quad (n = 3)$.

$\quad S^3(a\sqrt{3}/2) \times S^1(a/2)$, $\quad S^2(a/\sqrt{2}) \times S^2(a/\sqrt{2}) \quad (n = 4)$.

In $S^6(a)$ $\quad S^2(a/\sqrt{2}) \times S^1(a/2) \times S^1(a/2) \quad (n = 4)$,

$\quad S^4(2a/\sqrt{5}) \times S^1(a/\sqrt{5})$, $\quad S^3(a\sqrt{3/5}) \times S^2(a\sqrt{2/5}) \quad (n = 5)$.

In $S^7(a)$ $\quad S^1(a/2) \times S^1(a/2) \times S^1(a/2) \times S^1(a/2) \quad (n = 4)$,

$$S^3(a\sqrt{3/5}) \times S^2(a/\sqrt{5}) \times S^1(a/\sqrt{5}) ,$$
$$S^2(a\sqrt{2/5}) \times S^2(a\sqrt{2/5}) \times S^1(a/\sqrt{5}) \qquad (n = 5) ,$$
$$S^5(a\sqrt{5/6}) \times S^1(a/\sqrt{6}) , \quad S^4(a\sqrt{2/3}) \times S^2(a/\sqrt{3}) ,$$
$$S^3(a/\sqrt{2}) \times S^3(a/\sqrt{2}) \qquad (n = 6) .$$
$$\cdots\cdots\cdots\cdots$$

We now observe that in $S^m(a)$ no minimal submanifold of the type (5.2) is contained in an open semi-sphere, and shall show in Theorem 5.3 that this fact generally holds for any compact minimal submanifold in $S^m(a)$. We first need a lemma. Take a fixed unit vector e with components $(e^1, \cdots, e^{m+1})$ in R^{m+1}, and define a function ϕ in R^{m+1} by

$$\phi(x) = (x, e) = \sum_{A=1}^{m+1} x^A e^A , \qquad x \in R^{m+1} , \tag{5.4}$$

where $x = (x^1, \cdots, x^{m+1})$, and v denotes the restriction of ϕ to $S^m(a)$. Then along $S^m(a)$,

$$\nabla_i v = \sum_{A=1}^{m+1} B_i{}^A \nabla_A \phi ,$$

from which and (5.4) it follows that

$$\nabla_i v = \sum_{A=1}^{m+1} B_i{}^A e^A ,$$

and hence that

$$\nabla_j \nabla_i v = \sum_{A=1}^{m+1} (\nabla_j B_i{}^A) e^A = -v g_{ji}/a^2 ,$$

because along $S^m(a)$

$$\nabla_j B_i{}^A = -g_{ji} x^A / a^2 ,$$

Thus we have

Lemma 5.1. *In $S^m(a)$ there exists a nontrivial function v satisfying*

$$\nabla_j \nabla_i v = -v g_{ji}/a^2 , \tag{5.5}$$

where v is the restriction to $S^m(a)$ of the function ϕ defined in R^{m+1} by (5.4).

Next consider an n-dimensional minimal submanifold M^n in $S^m(a)$, $1 < n < m$. Then by transvecting (5.5) with $B_c{}^j B_b{}^i$ we have, along M^n,

$$B_c{}^j B_b{}^i \nabla_j \nabla_i v = v g_{cb}/a^2 , \tag{5.6}$$

which together with $\nabla_c B_b{}^h = H_{cb}{}^x C_x{}^h$ implies

$$\nabla_c \nabla_b v - H_{cb}{}^x C_x{}^i \nabla_i v = -v g_{cb}/a^2 .$$

Thus by transvecting with g^{cb} and the minimality of M^n we obtain

$$g^{cb}\nabla_c \nabla_b v = -nv/a^2 .$$

Since v cannot be positive (or negative) everywhere in a compact M^n, we have

Theorem 5.3. *If an n-dimensional submanifold M^n in an m-dimensional sphere S^m is compact and minimal $(1 < n < m)$, then in S^m there exists no open semi-sphere containing M^n. When the M^n is contained in a closed semi-sphere V of S^m, M^n lies on the boundary ∂V of V, which is a great sphere of S^m.*

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